Cuaderno de notas de trabajo

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Cuaderno 41

El Lagrangiano Logarítmico.

L ---- { Lagrangiano de un sistema dinámico.

$L(q_1, q_2, q_3 \cdots q_N; \dot{q}_1, \dot{q}_2, \cdots \dot{q}_N, t)$

$q_1, q_2, q_3 \cdots q_N$ --- { Coordenadas generalizadas del sistema.

Sea $\mathcal{L}$ el Lagrangiano Logarítmico. $\Longrightarrow$

$$L = e^{\mathcal{L}}.$$

$$\mathcal{L} = \mathcal{L}(q_1, q_2 \cdots q_N; \dot{q}_1, \dot{q}_2 \cdots \dot{q}_N, t).$$

Principio Variacional.

$$\delta \int_A^B e^{\mathcal{L}} dt = 0 .$$

Ecuaciones de Euler

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0}$$

$$\boxed{\frac{\partial L}{\partial \dot{q}_i} = e^{\mathcal{L}}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}}$$

$$\boxed{\frac{\partial L}{\partial q_i} = e^{\mathcal{L}}\frac{\partial \mathcal{L}}{\partial q_i}}$$

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = e^{\mathcal{L}}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) + e^{\mathcal{L}}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{d}{dt}\mathcal{L}}$$

Ecuaciones de Euler

$$\boxed{e^{\mathcal{L}}\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} + e^{\mathcal{L}}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{d}{dt}\mathcal{L} - e^{\mathcal{L}}\frac{\partial \mathcal{L}}{\partial q_i} = 0}$$

$$\boxed{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{d}{dt}\mathcal{L} - \frac{\partial \mathcal{L}}{\partial q_i} = 0}$$

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$$p_i = e^{\mathcal{L}} \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

Energía E

$$E = p_i \dot{q}_i - L$$

$$\cancel{E = e^{\mathcal{L}} \dot{q}_i - e}$$

$$\boxed{E = e^{\mathcal{L}} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i - e^{\mathcal{L}}}$$ Energía

Cuando $\frac{\partial \mathcal{L}}{\partial t} = 0$

$$\frac{dE}{dt} = e^{\mathcal{L}} \left[\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right.$$

$$\frac{dE}{dt} = \left[\frac{d}{dt}\left(e^{\mathcal{L}} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) \dot{q}_i + e^{\mathcal{L}} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \ddot{q}_i - e^{\mathcal{L}} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \ddot{q}_i - e^{\mathcal{L}} \frac{\partial \mathcal{L}}{\partial q_i} \dot{q}_i \right]$$

$$\frac{dE}{dt} = \left[e^{\mathcal{L}} \frac{\partial \mathcal{L}}{\partial q_i} \dot{q}_i + e^{\mathcal{L}} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \ddot{q}_i - e^{\mathcal{L}} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \ddot{q}_i - e^{\mathcal{L}} \frac{\partial \mathcal{L}}{\partial q_i} \dot{q}_i \right] = 0$$

Cuando $\frac{\partial \mathcal{L}}{\partial t} = 0 \Longrightarrow$

$$\frac{dE}{dt} = 0$$

$$\cancel{\frac{\partial \mathcal{L}}{\partial q_i} \dot{q}_i} \; \cancel{+} \; \cancel{\frac{\partial \mathcal{L}}{\partial q}}$$

$\cancel{\frac{d}{dt}}$

Cuando $\frac{\partial \mathcal{L}}{\partial t} = 0 \Longrightarrow$

$$\frac{d}{dt}\left[e^{\mathcal{L}} \left\langle \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i - 1 \right\rangle \right] = 0$$

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$$\boxed{e^{\mathcal{L}}\left\langle \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{q}_i - 1\right\rangle = \mathbf{E} = \text{Energía}}$$

Definición de Energía Logarítmica

$$\mathcal{E} = \log(-\mathbf{E})$$

$$\boxed{\mathcal{E} = \mathcal{L} + \log\left\langle 1 - \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right\rangle}$$

Caso Particular.
Campo Central.

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$$\cancel{\mathcal{L} = e^{\frac{2M}{r}}(1+v^2)}$$

$$\cancel{\frac{\partial \mathcal{L}}{\partial v}} \qquad \boxed{\mathcal{L} = \frac{2M}{r} + \log(1+v^2)}$$

$$\frac{\partial \mathcal{L}}{\partial \vec{v}} = \frac{2\vec{v}}{1+v^2}$$

$$\boxed{\vec{v}\cdot\frac{\partial \mathcal{L}}{\partial \vec{v}} = \frac{2v^2}{1+v^2}}$$

$$1 - \vec{v}\cdot\frac{\partial \mathcal{L}}{\partial \vec{v}} = \frac{1+v^2-2v^2}{1+v^2} = \frac{1-v^2}{1+v^2}$$

$$\boxed{1 - \vec{v}\cdot\frac{\partial \mathcal{L}}{\partial \vec{v}} = \frac{1-v^2}{1+v^2}}$$

$$\ell = \frac{2M}{r} + \log(1+v^2) + \log(1-v^2) - \log(1+v^2)$$

$$\boxed{\ell = \frac{2M}{r} + \log(1-v^2)}$$

Comprobación

Lagrangiano --- L;

Lagrangiano logarítmico --- $\mathcal{L}$;

$$L = e^{\mathcal{L}}.$$

Principio Variacional:

$$\delta \int_A^B e^{\mathcal{L}} dt = 0 .$$

$$\frac{\partial L}{\partial \dot{q}_i} = e^{\mathcal{L}} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} ;$$

$$\frac{\partial L}{\partial q_i} = e^{\mathcal{L}} \frac{\partial \mathcal{L}}{\partial q_i} .$$

Ecuaciones de Euler.

$$\frac{d}{dt}\left\{ e^{\mathcal{L}} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right\} - e^{\mathcal{L}} \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

$$\cancel{e^{\mathcal{L}}}\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\cancel{e^{\mathcal{L}}}\frac{d\mathcal{L}}{dt} - \cancel{e^{\mathcal{L}}}\frac{\partial \mathcal{L}}{\partial q_i} = 0$$

$$\cancel{\frac{d}{dt}\left\{ \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right\} - \frac{\partial \mathcal{L}}{\partial q_i} = 0}$$

$$\boxed{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{d\mathcal{L}}{dt} - \frac{\partial \mathcal{L}}{\partial q_i} = 0}$$

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Conservación de la Energía.

Hipótesis: $\frac{\partial \mathcal{L}}{\partial t} = 0$

$$q \quad \cancel{E} = \sum \dot{q}_i \frac{\partial \mathcal{L}}{}$$

$$\boxed{E = \sum \dot{q}_i \, e^{\mathcal{L}} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - e^{\mathcal{L}}}$$

Energía: $\boxed{E = e^{\mathcal{L}} \left\langle \sum \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - 1 \right\rangle}$

$$\frac{dE}{dt} = \left[e^{\mathcal{L}} \left\langle \sum \ddot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \sum \dot{q}_i \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right\rangle + e^{\mathcal{L}} \left\langle \frac{d\mathcal{L}}{dt} \sum \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{d\mathcal{L}}{dt} \right\rangle \right]$$

$$\frac{dE}{dt} = e^{\mathcal{L}} \left[\ddot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \dot{q}_i \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \left(\frac{d}{dt} \mathcal{L} \right) \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{d\mathcal{L}}{dt} \right]$$

$$\frac{dE}{dt} = e^{\mathcal{L}} \left[\ddot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \dot{q}_i \frac{\partial \mathcal{L}}{\partial q_i} - \frac{d\mathcal{L}}{dt} \right]$$

$$\frac{dE}{dt} = e^{\mathcal{L}} \left[\ddot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \dot{q}_i \frac{\partial \mathcal{L}}{\partial q_i} - \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \ddot{q}_i - \frac{\partial \mathcal{L}}{\partial q_i} \dot{q}_i - \frac{\partial \mathcal{L}}{\partial t} \right]$$

$$\boxed{\frac{dE}{dt} = -e^{\mathcal{L}} \frac{\partial \mathcal{L}}{\partial t}}$$

or

$$\boxed{\frac{dE}{dt} = -e^{\mathcal{L}} \frac{\partial \mathcal{L}}{\partial t}}$$

Si $\mathcal{L}$ es independiente de $t \Rightarrow$

$$\frac{dE}{dt} = 0$$

Definición de la Energía logarítmica:

$$\boxed{\mathcal{E} = \log(-E)}$$

$$\mathcal{E} = \log\left[e^{\mathcal{L}} \left\langle 1 - \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right\rangle\right]$$

$$\boxed{\mathcal{E} = \mathcal{L} + \log \left\langle 1 - \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right\rangle}$$

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$$\boxed{\mathcal{E} = \mathcal{L} + \log\left\langle 1 - \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right\rangle}$$

$$\mathcal{E} = \log(-E)$$

$$\frac{d\mathcal{E}}{dt} = \frac{-\frac{dE}{dt}}{-E} = \frac{\dot{E}}{E}$$

Si E es constante $\Longrightarrow$

$$\boxed{\frac{d\mathcal{E}}{dt} = 0}$$

$$\cancel{\mathcal{E} = \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L}$$

$$\boxed{E = \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L}$$

$$\frac{dE}{dt} = \left[\cancel{\ddot{q}_i \frac{\partial L}{\partial \dot{q}_i}} + \dot{q}_i \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}\dot{q}_i - \cancel{\frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i} - \frac{\partial L}{\partial t} \right]$$

$$\boxed{\frac{dE}{dt} = -\frac{\partial L}{\partial t}}$$

$$\mathcal{E} = \log(-E)$$

$$e^{\mathcal{E}} = -E$$

$$e^{\mathcal{E}}\frac{d\mathcal{E}}{dt} = +\frac{\partial}{\partial t}e^{\mathcal{L}} = + e^{\mathcal{L}}\frac{\partial \mathcal{L}}{\partial t}$$

$$e^{\mathcal{E}-\mathcal{L}}\frac{d\mathcal{E}}{dt} = \frac{\partial \mathcal{L}}{\partial t}$$

$$\left\langle 1 - \dot{q}_i\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right\rangle \frac{d\mathcal{E}}{dt} = \frac{\partial \mathcal{L}}{\partial t}$$